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Non-linear oscillations of fluid in a container

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This paper is concerned with forced oscillations of fluid in a rectangular container. From the linearized approximation of the equations governing these oscillations, resonance frequencies are obtained for which the amplitude of the oscillations becomes infinite. Observation shows that under these circumstances a hydraulic jump is formed, which travels periodically back and forth between the walls of the container. This hydraulic jump is a non-linear phenomenon, analogous to the shock wave appearing in one-dimensional gas flow under similar resonance conditions.

A theory developed by previous authors for one-dimensional gas flow is applied to the fluid oscillations in order to calculate the strength and the phase of the jump. The moment exerted on the container is also calculated. These quantities were measured experimentally at the lowest resonance frequency and the results are in good agreement with the theoretical values.

1. Introduction

When a rectangular vessel containing fluid is oscillated at small amplitudes about a fixed axis (figure 1), gravity waves appear on the surface of the fluid. If the width B of the container is large with respect to the depth h_0 of the fluid, the wave height η might be described by the 'linear shallow-water theory'. In § 2 it will be shown that according to this theory the amplitude of the surface elevation is proportional to $\{\cos(\pi\omega/2\omega_0)\}^{-1}$, where ω is the angular frequency of the excitation and

$$\omega_0 = (\pi/B)(gh_0)^{\frac{1}{2}}. \quad (1)$$

Hence the linear theory predicts an infinite amplitude at $\omega = \omega_0$. The present paper is concerned with the frequency range where ω is near ω_0.

Experiments carried out with $\omega \approx \omega_0$ showed the occurrence of a hydraulic jump which travelled back and forth between the walls of the container. Obviously the linear theory is invalid in this frequency range and a description must start from the non-linear shallow-water theory.

The present situation appears to be analogous to that occurring when a column of gas is oscillated at a resonance frequency, in which case a shock wave is formed in the gas. This problem has received attention recently in the work of Betchov (1958), Chu & Ying (1963) and Chester (1964). The most rigorous account of the travelling shock wave appearing in the gas was given by Chu & Ying, who used a perturbation method due to Lin (1954). In the present paper the Chu–Ying–Lin method is applied to fluid oscillations under resonance conditions.

Following a formulation of the problem in §2, the theory is discussed in §3 and results pertaining to the hydraulic jump are given in §4. Experimental results are presented in §5 together with the theoretical results.

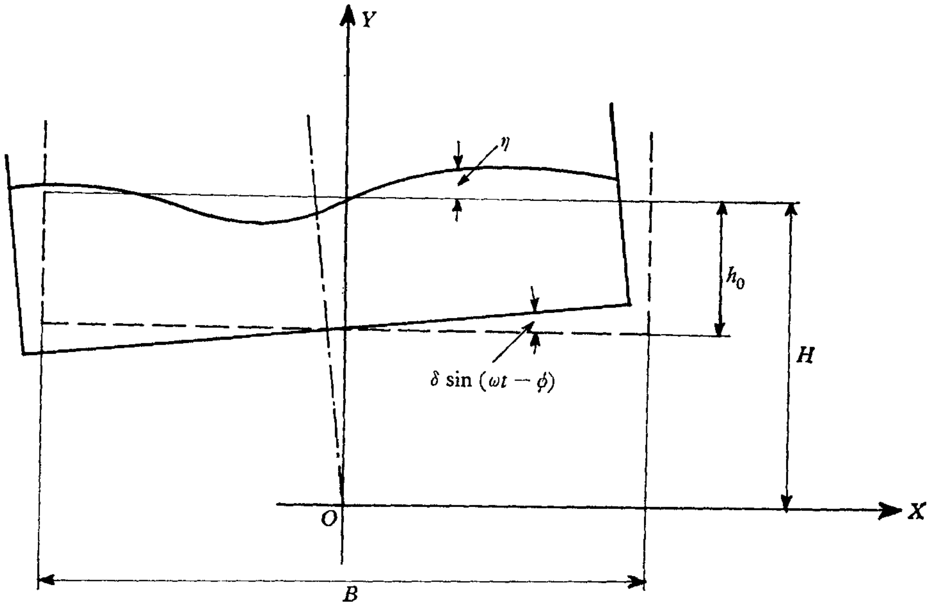

FIGURE 1. Fluid oscillating in a container.

2. Theoretical analysis

Consider an open rectangular container of width B, filled with fluid to a level h_0 (see figure 1). Let one pair of the side walls be parallel to the (X, Y)-plane, where gravity acts in the negative Y-direction. Let the other side walls be located at $x = \pm \frac{1}{2}B$. The container is oscillated about the Z-axis at small amplitudes δ.

We assume that the width of the container in the Z-direction is large enough for the flow to be two-dimensional. We denote the undisturbed fluid surface by $y = H$, the surface elevation with respect to this level by η, and the angular displacement about O by $\delta \sin \omega t$, a counter-clockwise rotation being considered positive. Then the bottom is described by

$$y = H - h_0 + \delta x \sin \omega t, \tag{2}$$

and the surface of the fluid by $$y = H + \eta. \tag{3}$$

It is convenient to consider the surface level relative to the bottom of the container. Therefore we introduce

$$\lambda = h_0 + \eta - \delta x \sin \omega t. \tag{4}$$

If $h_0/B \ll 1$, the motion of the fluid caused by the oscillation of the container can be described by the 'shallow-water theory' (Wehausen & Laitone 1960, §30, Stoker 1957, ch. 2). In this theory the continuity equation is

$$\frac{\partial \lambda}{\partial t} + u \frac{\partial \lambda}{\partial x} + \lambda \frac{\partial u}{\partial x} = 0, \tag{5}$$

where u denotes the velocity in the x-direction, and the momentum equation is

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+g\frac{\partial \lambda}{\partial x}+g\,\delta\sin\omega t = 0. \tag{6}$$

In this formulation the pressure distribution in the vertical direction is assumed to be hydrostatic. Therefore, the acceleration in the Y-direction, introduced by the excitation, must be small with respect to the acceleration due to gravity; i.e. $\delta B\omega^2/g \ll 1$. The boundary conditions for u are determined by the velocity produced in the horizontal direction by the exciting oscillation. In the shallow-water approximation u does not vary between the bottom and the surface. Taking the value at the surface, we require that

$$u = -\delta H\omega\cos\omega t \quad \text{at} \quad x = \pm\tfrac{1}{2}B. \tag{7}$$

We seek a solution of equations (5)–(7) in which u and λ vary periodically. For small enough δ, one expects the linearized form of (5) and (6) to be valid, i.e.

$$\frac{\partial \lambda}{\partial t}+h_0\frac{\partial u}{\partial x} = 0,$$

$$\frac{\partial u}{\partial t}+g\frac{\partial \lambda}{\partial x}+g\delta\sin\omega t = 0.$$

The solution of these equations for λ, satisfying (7), is

$$\lambda = h_0-\frac{\delta B\omega_0}{\pi\omega}\,\frac{1+H\omega^2/g}{\cos(\pi\omega/2\omega_0)}\sin\omega t\sin\frac{\pi\omega x}{B\omega_0}, \tag{8}$$

where ω_0 is defined by (1).

However, for $\omega \to \omega_0$, equation (8) gives $\lambda \to \infty$. Experiments described in §5 showed the appearance of a hydraulic jump or bore for $\omega \approx \omega_0$. Obviously, the linearized equations do not hold under these circumstances and a description therefore has to start from equations (5) and (6). The situation is analogous to that in gas dynamics when a column of gas is oscillated at small amplitude, e.g. by a piston (see figure 2).

In terms of the density ρ and velocity u, the acoustic approximation is

$$\frac{\partial \rho}{\partial t}+\rho_0\frac{\partial u}{\partial x} = 0,$$

$$\frac{\partial u}{\partial t}+\frac{a^2}{\rho_0}\frac{\partial \rho}{\partial x} = 0,$$

where the undisturbed quantities are indicated with the subscript 0, and a is the velocity of sound in the gas. If $u = 0$ at the closed end $x = L$, and at the piston $u = k\omega\cos\omega t$, the solution to the above equations is

$$u = k\omega\cos\omega t\,\frac{\sin\{\omega(L-x)/a\}}{\sin(\omega L/a)}. \tag{9}$$

Resonance occurs when $\omega L/a$ is a multiple of π. Then a shock wave is generated in the gas, which travels periodically to and fro through the column, similar to the hydraulic jump described in this paper. Indeed it is well known that the equations of shallow-water theory are equivalent to the equations of one-dimensional gas dynamics.

The problem of resonance oscillations in a gas column was treated by Betchov (1958), Chester (1964) and Chu & Ying (1963), starting from the assumption that the excitation amplitudes were sufficiently small to permit a linearized solution of type (9) for conditions far from resonance. All these authors succeeded in obtaining approximate solutions at resonance, including the case of shock waves travelling periodically up and down through the gas.

Betchov (1958) and Chester (1964) derived such a solution by both physical and mathematical arguments and also discussed the influence of viscosity.

Chu & Ying used a method of characteristics perturbation developed by Lin (1954), which, if properly adapted, appears to be applicable to the hydraulic-jump problem. We shall give a brief outline of the method, referring for details to the work of Chu & Ying, henceforth denoted by C.Y.

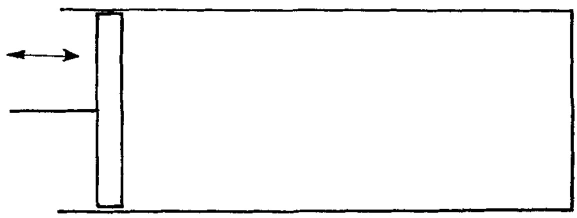

FIGURE 2. Gas-dynamic analogy: oscillations of a gas column excited by a piston.

3. The Chu–Ying–Lin method as applied to the hydraulic jump problem

We introduce
$$c = (g\lambda)^{\frac{1}{2}}, \tag{10}$$
$$c_0 = (gh_0)^{\frac{1}{2}}, \tag{11}$$
and
$$\epsilon^2 = g\delta/\omega_0 c_0,$$
or in view of (1),
$$\epsilon = (B\delta/\pi h_0)^{\frac{1}{2}}. \tag{12}$$

Further, we allow for a shift in time by introducing an additional phase ϕ in the motion of the bottom of the container. Using (10)–(12), we obtain from (5) and (6)
$$\left\{\frac{\partial}{\partial t}+(u+c)\frac{\partial}{\partial x}\right\}\left\{u+2c-\frac{\omega_0}{\omega}\epsilon^2 c_0\cos(\omega t-\phi)\right\}=0, \tag{13}$$
$$\left\{\frac{\partial}{\partial t}+(u-c)\frac{\partial}{\partial x}\right\}\left\{u-2c-\frac{\omega_0}{\omega}\epsilon^2 c_0\cos(\omega t-\phi)\right\}=0. \tag{14}$$

We define the characteristic co-ordinates α and β such that along the C^+-characteristics given by
$$\frac{\partial x}{\partial \alpha}=(u+c)\frac{\partial t}{\partial \alpha}, \tag{15}$$
β is constant, and along the C^--characteristics given by
$$\frac{\partial x}{\partial \beta}=(u-c)\frac{\partial t}{\partial \beta}, \tag{16}$$

α is constant. Then it follows from (13) and (14) that along C^+

$$\frac{\partial}{\partial\alpha}\left\{u+2c-\frac{\omega_0}{\omega}\epsilon^2 c_0\cos(\omega t-\phi)\right\}=0, \tag{17}$$

and along C^-

$$\frac{\partial}{\partial\beta}\left\{u-2c-\frac{\omega_0}{\omega}\epsilon^2 c_0\cos(\omega t-\phi)\right\}=0. \tag{18}$$

Note that the Riemann invariants, i.e. the expressions $\{\}$ in (17) and (18), assume a simple form involving only u and c, owing to the fact that the bottom slope does not depend on x. A periodic solution of equations (15)–(18), involving hydraulic jumps and satisfying the boundary condition (7) has to be found. A possible approach to the problem might be to start from the undisturbed conditions and to construct the development of the flow by the method of characteristics. This development would lead to the formation of a hydraulic jump. Using the methods given in Courant & Friedrichs (1948) for the gas-dynamic case, the procedure might be continued till a quasi-stationary situation is reached, in which a jump travels periodically to and fro. Such an approach would be conceivable with the aid of a computer.

Another approach is to start from the concept of the quasi-stationary situation (mentioned above) and to attempt an analytical construction of such a solution.

Consider the (x, t)-plane (figure 3). The paths of the jumps are represented by PQ, QR, RS and so on, and the aforementioned solution has to satisfy equations (15)–(18) in the different regions I, II, III, etc. The solutions for these regions, which have different energies, must be related to each other by the jump conditions, which require the conservation of mass and momentum across the hydraulic jump. (An important difference occurs with shock waves, because across a shock wave energy is preserved.) From consideration of the periodicity it follows that the flow in region I must be repeated in region III, and so on. The difficulty is that the paths of the jumps are not known at the outset. We know that for $\delta\to 0$, these paths cannot be far from $dx/dt=\pm c_0$. The deviations of the jumps from these directions are most conveniently expressed as perturbations in terms of the characteristic co-ordinates (Lin 1954), the appropriate perturbation parameter ϵ being defined by (12). For, if the difference in level across the jump is $\Delta\lambda$, then the rate of loss of energy is given by†

$$\frac{dE}{dt}\approx\frac{(\Delta\lambda)^3}{h_0^2}Bh_0\omega\rho g,$$

while the work W done by the external forces is

$$W\approx\Delta\lambda B^2\delta\omega\rho g.$$

Equating these expressions yields

$$\Delta\lambda/h_0\approx(\delta B/h_0)^{\frac{1}{2}}\approx\epsilon. \tag{19}$$

The strength of the jump is thus of order ϵ.

We now write

$$u=\epsilon u_1(\alpha,\beta)+\epsilon^2u_2(\alpha,\beta)+\ldots, \tag{20}$$

$$c=c_0+\epsilon c_1(\alpha,\beta)+\epsilon^2c_2(\alpha,\beta)+\ldots, \tag{21}$$

$$x=x_0(\alpha,\beta)+\epsilon x_1(\alpha,\beta)+\epsilon^2x_2(\alpha,\beta)+\ldots, \tag{22}$$

$$t=t_0(\alpha,\beta)+\epsilon t_1(\alpha,\beta)+\epsilon^2t_2(\alpha,\beta)+\ldots. \tag{23}$$

† In C.Y. a similar argument is given for the gas-dynamic case.

In the course of the analysis the period of the jump, i.e. the time needed by the jump to travel once back and forth in the container, is also expanded in a series. For this reason the associated frequency ω is written as

$$\omega = \omega_0 + \epsilon\omega_1 + \dots .$$

In C.Y. this step is postponed till the last stage of the analysis after results have been obtained from equations with terms of order ϵ in which ω is treated as a constant. This inconsistency leads to an incorrect result for ϕ.

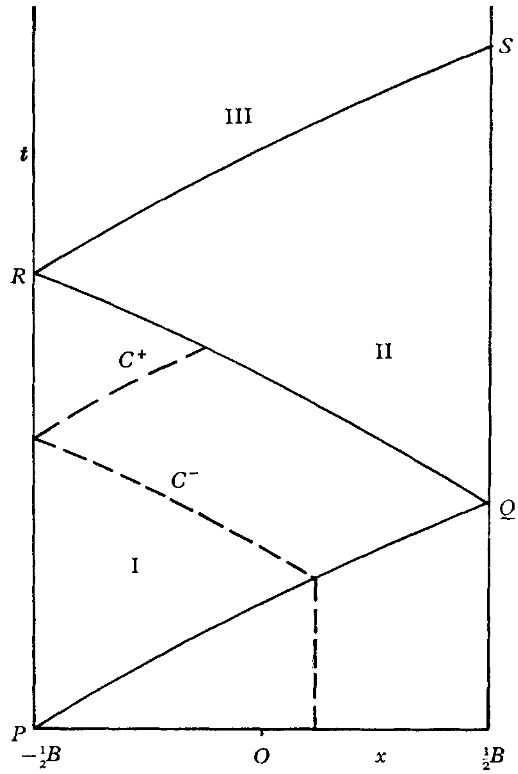

FIGURE 3. Paths of hydraulic jumps in (x, t)-plane.

Substituting these expressions into the characteristic equations (15)–(18) and collecting terms of like order in ϵ, we obtain

$$\epsilon^0: \quad \frac{\partial x_0}{\partial \alpha} = c_0 \frac{\partial t_0}{\partial \alpha}; \quad \frac{\partial x_0}{\partial \beta} = -c_0 \frac{\partial t_0}{\partial \beta}; \tag{24}$$

$$\left.\begin{aligned} \epsilon^1: \quad & \frac{\partial x_1}{\partial \alpha} = c_0 \frac{\partial t_1}{\partial \alpha} + (u_1 + c_1) \frac{\partial t_0}{\partial \alpha}, \\ & \frac{\partial x_1}{\partial \beta} = c_0 \frac{\partial t_1}{\partial \beta} + (u_1 - c_1) \frac{\partial t_0}{\partial \beta}, \\ & \frac{\partial u_1}{\partial \alpha} = -2 \frac{\partial c_1}{\partial \alpha}, \; \frac{\partial u_1}{\partial \beta} = 2 \frac{\partial c_1}{\partial \beta}, \end{aligned}\right\} \tag{25}$$

$$\left.\begin{aligned}\epsilon^2:\quad \frac{\partial x_2}{\partial \alpha} &= c_0\frac{\partial t_2}{\partial \alpha}+(u_1+c_1)\frac{\partial t_1}{\partial \alpha}+(u_2+c_2)\frac{\partial t_0}{\partial \alpha},\\ \frac{\partial x_2}{\partial \beta} &= -c_0\frac{\partial t_2}{\partial \beta}+(u_1-c_1)\frac{\partial t_1}{\partial \beta}+(u_2-c_2)\frac{\partial t_0}{\partial \beta},\\ \frac{\partial u_2}{\partial \alpha} &= -2\frac{\partial c_2}{\partial \alpha}+c_0\frac{\partial}{\partial \alpha}\{\cos(\omega_0 t_0-\phi)\},\\ \frac{\partial u_2}{\partial \beta} &= 2\frac{\partial c_2}{\partial \beta}+c_0\frac{\partial}{\partial \beta}\{\cos(\omega_0 t_0-\phi)\}.\end{aligned}\right\} \tag{26}$$

To solve these equations in region I (figure 3), we have first to formulate the conditions at the boundaries of this region in terms of α and β.

In defining α and β we follow C.Y. The C^--characteristics (α = const.) are defined by the value of x at the intersection with the hydraulic jump PQ. PQ is thus determined by $\alpha = x$. The C^+-characteristics (β = const.) are defined in the following manner: through each point of the wall PR passes a C^+-characteristic. The value of β along this characteristic is equal to the value of α along the C^--characteristic through the same point of PR. By this definition of the C^+-lines, the wall PR is described by $\alpha = \beta$.

Along $\alpha = \beta$ we have two conditions

$$u = -H\omega\delta\cos(\omega t-\phi), \tag{27}$$

and

$$x = -\tfrac{1}{2}B. \tag{28}$$

The hydraulic jump PQ tends to $\beta = -\frac{1}{2}B$ as $\epsilon \to 0$. Therefore, along PQ,

$$\beta = -\tfrac{1}{2}B+\epsilon\mu_1(\alpha)+\epsilon^2\mu_2(\alpha)+\ldots, \tag{29}$$

where $\mu_1(-\frac{1}{2}B) = \mu_2(-B\frac{1}{2}) = 0$, since at P, $\beta = \alpha = -\frac{1}{2}B$. Along (29) we have the condition $x = \alpha$.

The second condition that is needed is provided by stipulating that along $x = \alpha$

$$c = c_0+\epsilon\gamma_1(\alpha)+\epsilon^2\gamma_2(\alpha)+\ldots. \tag{30}$$

The functions γ_1 and γ_2 are determined later.

Finally, we fix the origin of time t at $\alpha = \beta = -\frac{1}{2}B$, so that at $t = 0$ the jump is at $x = -\frac{1}{2}B$. The boundary conditions are also determined in terms of ϵ.

Making use of Taylor expansions, we obtain

$$\begin{aligned}\epsilon^0:\quad & x_0 = -\tfrac{1}{2}B \quad \text{at} \quad \beta = \alpha,\\ & x_0 = \alpha \quad \text{at} \quad \beta = -\tfrac{1}{2}B,\\ & t_0 = 0 \quad \text{at} \quad \beta = \alpha = -\tfrac{1}{2}B;\\ \epsilon^1:\quad & x_1 = 0, \quad u_1 = 0 \quad \text{at} \quad \beta = \alpha,\\ & x_1+\mu_1\frac{\partial x_0}{\partial \beta} = 0, \quad c_1 = \gamma_1 \quad \text{at} \quad \beta = -\tfrac{1}{2}B;\\ & t_1 = 0 \quad \text{at} \quad \beta = \alpha = -\tfrac{1}{2}B;\\ \epsilon^2:\quad & x_2 = 0, \quad u_2 = -\pi^2 Hhc_0B^{-2}\cos(\omega_0 t_0-\phi) \quad \text{at} \quad \beta = \alpha,\\ & x_2+\mu_1\frac{\partial x_1}{\partial \beta}+\tfrac{1}{2}\mu_1^2\frac{\partial^2 x_0}{\partial \beta^2}+\mu_2\frac{\partial x_0}{\partial \beta} = 0, \quad c_2+\mu_1\frac{\partial c_1}{\partial \beta} = \gamma_2 \quad \text{at} \quad \beta = -\tfrac{1}{2}B\\ & t_2 = 0 \quad \text{at} \quad \beta = \alpha = -\tfrac{1}{2}B.\end{aligned}$$

Following C.Y., the procedure was as follows.

First, the solution in region I (figure 3) was obtained by solving the differential equations (24)–(26) with the above conditions. From this solution the flow in region I along QR, defined similarly to that along PQ (cf. equation (29)) as $\alpha = \frac{1}{2}B+\epsilon\theta_1(\beta)+\ldots$, was determined. The flow along QR in region II could be obtained subsequently from the conditions of conservation of mass and momentum across a jump. If the quantities in front of the jump are denoted by the subscript f, and behind the jump by b, these conditions are as follows:

$$(u_f-V_s)\,c_f^2 = (u_b-V_s)\,c_b^2,$$
$$(u_f-V_s)^2\,c_f^2+\tfrac{1}{2}c_f^4 = (u_b-V_s)^2\,c_b^2+\tfrac{1}{2}c_b^4,$$

where V_s is the velocity of propagation of the jump.

In the case of weak jumps, it follows that

$$u_f+2c_f = u_b+2c_b+O\{(c_b-c_f)^3\}, \tag{31}$$
$$V_s = u_f-c_f-\tfrac{3}{2}(c_b-c_f)-\tfrac{5}{8}(c_b-c_f)^2+O\{(c_b-c_f)^3\}. \tag{32}$$

Equations (31) and (32), were used to determine the flow along QR in region II. (Note that V_s is not a new unknown, since $V_s = (dx/dt)$ along the jump.)

The next step was to calculate u, c, x and t in region II, using differential equations analogous to (24)–(26), the boundary conditions along QS, and knowledge of the flow along QR. The solution for region II was used to determine the flow along RS, from which the flow in region III along the jump RS was obtained by application of the jump conditions. From the periodicity requirement, the flow along RS in region III must be a repetition of the flow in region I along PQ. In particular, the distribution of c along PQ, represented by (30), must be the same as the distribution of c along RS. This condition yielded differential equations for the functions γ_1 and γ_2. The solution for γ_1 provided, to second order in ϵ, the flow variables u and c in both region I and region II, as well as the paths of the jumps. The procedure involves a long series of calculations and we have indicated, following C.Y., how to start the calculations in region I and have outlined how final results were obtained.

Presentation of the full-length calculation, would require an undue amount of space, and therefore only the results are given. We refer the reader for the details to C.Y., where the method is applied to the oscillations of a gas column at resonance.

The present authors found the calculation of the phase of the jump in C.Y. to be incorrect, due to inconsistency in the expansion of the frequency, which has been mentioned earlier in this section. Therefore we did not follow C.Y. in calculating the phase of the jump, but used a method discussed in the next section.

4. Results

Using the method of the preceding sections, we obtained the following results (reference being made to figure 3):

$$u^{\mathrm{I}} = 4\epsilon c_0 A \sin\tfrac{1}{2}(\omega_0 t-\phi-\tfrac{1}{2}\pi)\sin\frac{\pi(x+\frac{1}{2}B)}{2B}+O(\epsilon^2), \tag{33}$$

$$c^{\mathrm{I}} = c_0+\epsilon\left[\frac{B\Omega}{\pi}+\frac{2c_0A}{\pi}\sin\left(\tfrac{1}{2}\phi+\tfrac{1}{4}\pi\right)\right.$$
$$\left.+2c_0A\cos\tfrac{1}{2}(\omega_0t-\phi-\tfrac{1}{2}\pi)\cos\frac{\pi(x+\frac{1}{2}B)}{2B}\right]+O(\epsilon^2), \tag{34}$$

where
$$\Omega = (\omega-\omega_0)/\epsilon, \tag{35}$$

and
$$A = \left[\frac{2}{3\pi}\left(1+\frac{\pi^2h_0H}{B^2}\right)\right]^{\frac{1}{2}} \tag{36}$$

In region II,

$$u^{\mathrm{II}} = 4\epsilon c_0A\cos\tfrac{1}{2}(\omega_0t-\phi-\tfrac{1}{2}\pi)\cos\frac{\pi(x+\frac{1}{2}B)}{2B}+O(\epsilon^2), \tag{37}$$

$$c^{\mathrm{II}} = c_0+\epsilon\left[\frac{B\Omega}{\pi}+\frac{2c_0A}{\pi}\sin\left(\tfrac{1}{2}\phi+\tfrac{1}{4}\pi\right)\right.$$
$$\left.+2c_0A\sin\tfrac{1}{2}(\omega_0t-\phi-\tfrac{1}{2}\pi)\sin\frac{\pi(x+\frac{1}{2}B)}{2B}\right]+O(\epsilon)^2. \tag{38}$$

The path of the jump PQ, travelling from left to right, appears to be

$$x = -\tfrac{1}{2}B+c_0t+\epsilon\left[\frac{B\Omega}{\pi}t+\frac{2A}{\pi}(c_0t-B)\sin\left(\tfrac{1}{2}\phi+\tfrac{1}{4}\pi\right)\right.$$
$$\left.-2\frac{AB}{\pi}\sin\left(\tfrac{1}{2}\omega_0t-\tfrac{1}{2}\phi-\tfrac{1}{4}\pi\right)\cos\tfrac{1}{2}\omega_0t\right]+O(\epsilon^2), \tag{39}$$

while the equation for the jump QR, travelling from right to left is

$$x = \frac{3B}{2}-c_0t-\epsilon\left[\frac{B\Omega}{\pi}t+\frac{2A}{\pi}(c_0t-2B)\sin\left(\tfrac{1}{2}\phi+\tfrac{1}{4}\pi\right)\right.$$
$$\left.+\frac{2BA}{\pi}\cos\left(\tfrac{1}{2}\omega_0t-\tfrac{1}{2}\phi-\tfrac{1}{2}\pi\right)\sin\tfrac{1}{2}\omega_0t\right]+O(\epsilon^2). \tag{40}$$

The phase difference ϕ between the jump and the container is determined from the condition that at all times the total water volume per unit length in the z-direction should be Bh_0, i.e.

$$\int_{-\frac{1}{2}B}^{\frac{1}{2}B}\lambda\,dx = Bh_0,$$

or, by use of (10) and (11),

$$\int_{-\frac{1}{2}B}^{\frac{1}{2}B}c^2dx = Bc_0^2.$$

The integral can be evaluated with the help of (34) and (38). To obtain the contribution of order ϵ, the relation between x and t at the jump is needed only to zeroth order. If we consider a time t' at which the jump travels from the left to the right, it follows from (39) to this order that at the jump,

$$x = -\tfrac{1}{2}B+c_0t'.$$

Therefore we have to use (34) from $x = -\frac{1}{2}B$ to $x = -\frac{1}{2}B+c_0t'$. For the remainder of the integration interval, i.e. the region in front of the jump, we have to use (38), with (see figure 3) $t'+2B/c_0$ substituted for t'. Then we obtain, requiring the cancellation of terms of order ϵ,

$$\sin\left(\tfrac{1}{2}\phi+\tfrac{1}{4}\pi\right) = -B\Omega/6Ac_0+O(\epsilon). \tag{41}$$

For $\omega = \omega_0$, (41) with (35) yields $\phi = -\frac{1}{2}\pi$. This means that when the container starts an oscillation cycle (in the counter-clockwise direction) with $\omega = \omega_0$, the jump is, on its way from $x = \frac{1}{2}B$ to $x = -\frac{1}{2}B$, just passing the centre of the container and is therefore exactly 90° out of phase with the excitation.

As a check on the expressions for the paths of the jump, we calculated the mean speed

$$\frac{\omega}{\pi}\int_0^{\pi/\omega} \frac{dx}{dt}\,dt$$

from (39). The result obtained is $c_0 + \epsilon\Omega B/\pi$ or, by virtue of (35), $\omega B/\pi$ as it should be.

We now determine the strength of the hydraulic jump, i.e. the difference $[\eta]$ between the surface elevations behind and in front of the jump, divided by h_0. From (4), (10) and (11) we infer that

$$c = c_0(1 + \eta/2h_0) + O(\epsilon^2). \tag{42}$$

From (42) we obtain, using (34), (35), (38) and (41), for the wave elevations in region I and region II

$$\frac{\eta^{\mathrm{I}}}{h_0} = \frac{4}{3}\frac{\omega - \omega_0}{\omega_0} + 4A\epsilon\cos\left(\tfrac{1}{2}\omega_0 t - \tfrac{1}{2}\phi - \tfrac{1}{4}\pi\right)\cos\frac{\pi(x + \frac{1}{2}B)}{2B} + O(\epsilon^2), \tag{43}$$

$$\frac{\eta^{\mathrm{II}}}{h_0} = \frac{4}{3}\frac{\omega - \omega_0}{\omega_0} + 4A\epsilon\sin\left(\tfrac{1}{2}\omega_0 t - \tfrac{1}{2}\phi - \tfrac{1}{4}\pi\right)\sin\frac{\pi(x + \frac{1}{2}B)}{2B} + O(\epsilon^2). \tag{44}$$

Now consider again a time t' at which the jump travels from the left to the right. The position of the jump is given by (39), and the strength of the jump follows from

$$\begin{aligned}\frac{[\eta]}{h_0} &= \frac{1}{h_0}\Big[\eta^{\mathrm{I}} - \tfrac{1}{2}B + c_0 t' + O(\epsilon), t' \\ &\qquad - \eta^{\mathrm{II}}\Big(-\tfrac{1}{2}B + c_0 t' + O(\epsilon), t' + \frac{2B}{c_0} + O(\epsilon)\Big)\Big] \\ &= 4A\epsilon\cos\left(\tfrac{1}{2}\phi + \tfrac{1}{4}\pi\right).\end{aligned}$$

Using (41), we obtain

$$[\eta]/h_0 = 4A\epsilon\{1 - (B\Omega/6Ac_0)^2\}^{\frac{1}{2}} + O(\epsilon^2). \tag{45}$$

For a given frequency the strength of the jump is independent of time to second order in ϵ.

5. Measurements

The theoretical results given in the preceding section were verified experimentally. For this purpose a rectangular container with $B = 1{\cdot}20\,\mathrm{m}$, was filled with tap water to a level $h_0 = 9 \times 10^{-2}\,\mathrm{m}$, and oscillated about an axis parallel to the Z-axis as in figure 1 and coinciding with the bottom of the container. The experimental value of H was therefore equal to h_0 and hence the second term in (36) could be omitted, being negligible with respect to unity.

Three different kinds of measurements were carried out:

(i) At an oscillation amplitude $\delta = \pi/90$ rad., the surface elevation was measured at four different values of x, the frequency being $\omega_0 = 2{\cdot}46\,\mathrm{sec}^{-1}$.

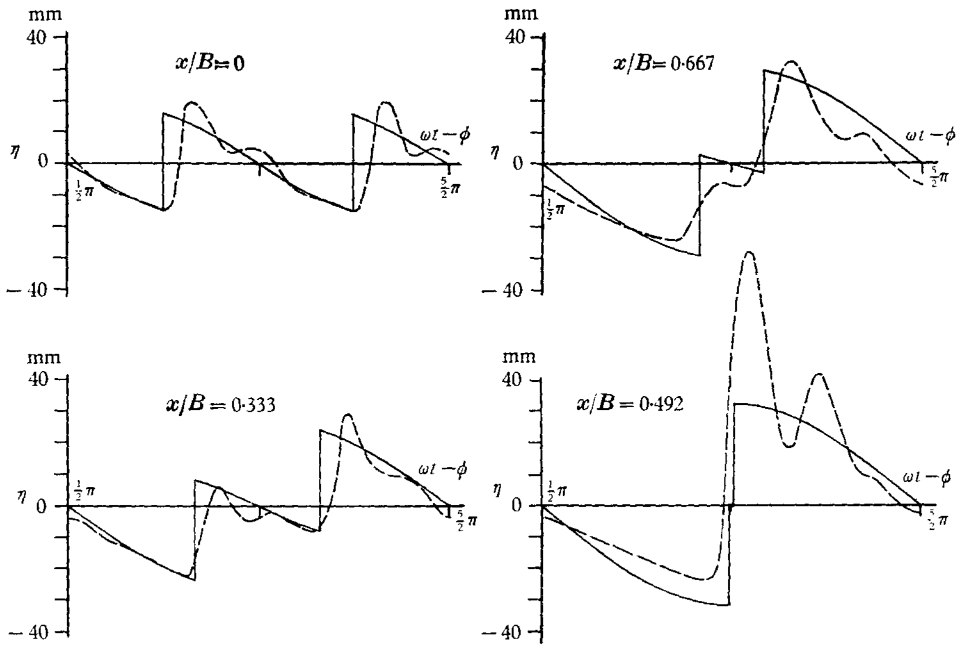

FIGURE 4. Surface elevation as a function of $\omega t - \phi$ for various values of x, at $\omega = \omega_0$ and $\delta = 2°$. ———, Theory; – – – –, experiment.

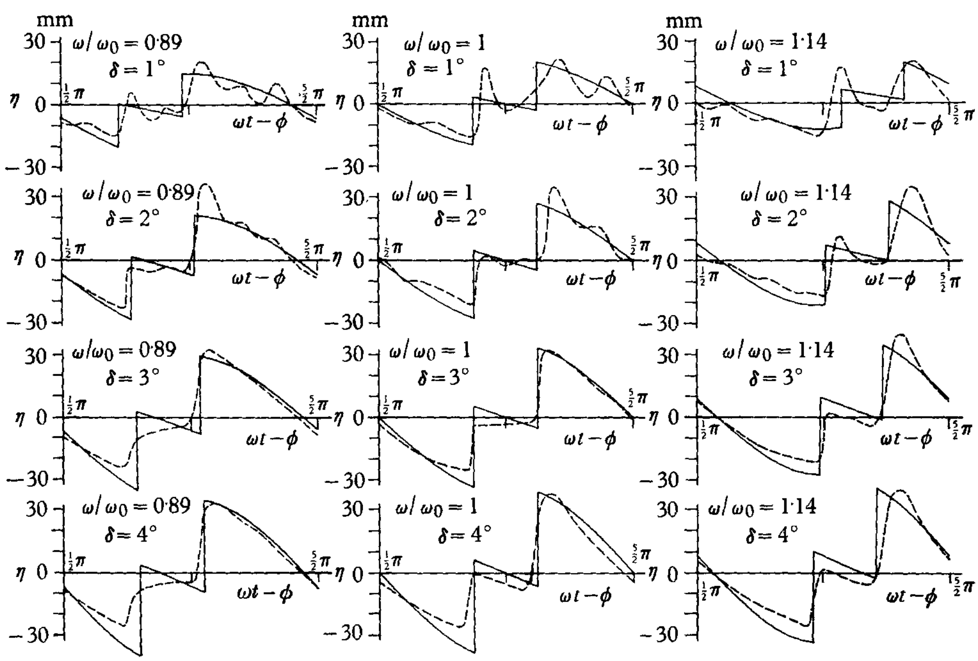

FIGURE 5. Surface elevation as a function of $\omega t - \phi$ at $x = \frac{1}{4}B$, for various values of δ and ω. ———, Theory; – – – –, experiment.

The experimental results are given in figure 4 as broken lines. The corresponding values obtained from the theoretical results given in §4 are represented by solid lines.

(ii) At a fixed value of x, the surface elevation η was measured for various values of the oscillation amplitude, viz. $\delta = \pi/180, \pi/90, \pi/60, \pi/45$ rad. and for

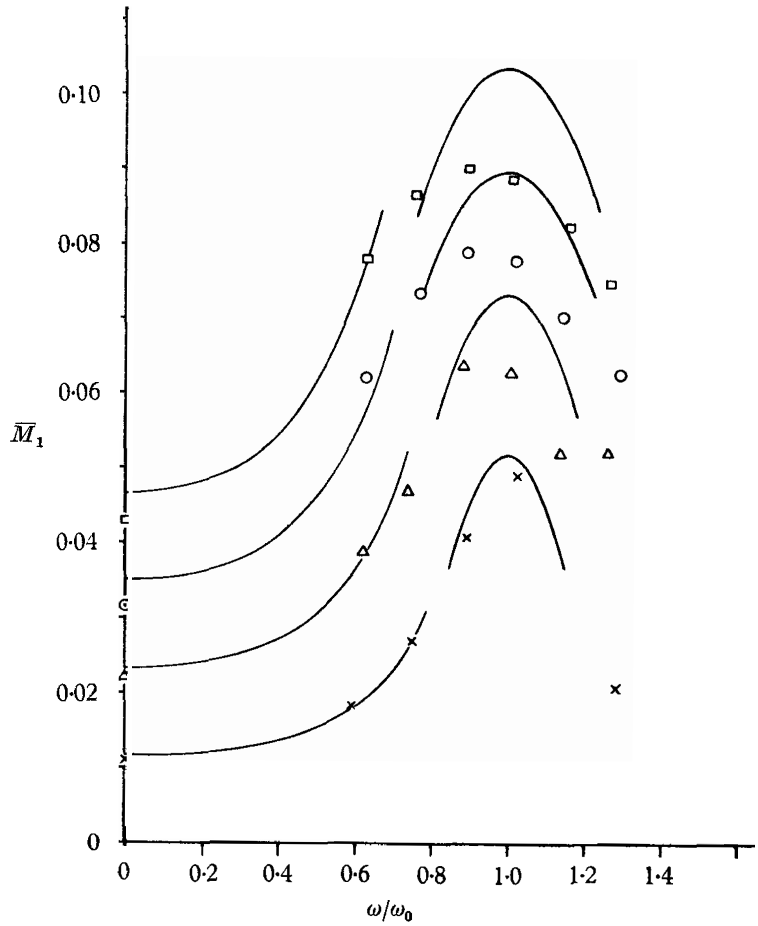

FIGURE 6. Amplitude of the moment exerted by the fluid on the container as a function of ω/ω_0 for various values of δ. ———, Theory (for low values of ω/ω_0, the curve obtained from linearized theory is drawn).

$\omega = 0{\cdot}89\,\omega_0$, $\omega = \omega_0$, and $\omega = 1{\cdot}14\omega_0$. The results are given in figure 5 in broken lines and the corresponding theoretical results are represented by solid lines.

(iii) The moment about O exerted by the fluid was measured as a function of ω for several values of the oscillation amplitude δ. A counter-clockwise moment was considered positive.

The measured moment was subjected to a Fourier analysis

$$\frac{M}{\rho g(\frac{1}{2}B)^3} = \sum_{n=1}^{\infty} \overline{M}_n \sin\{n(\omega_0 t - \phi) - \psi_n\}. \tag{46}$$

We recall in this connexion that $\omega_0 t - \phi$ is the phase of the container (since in the present analysis $\omega - \omega_0 = O(\epsilon)$), so that ψ_n measures the phase difference between the nth harmonic in M and the nth harmonic of the container. The experimental values of $\overline{M}_1$, for several values of δ, are given in figure 6 as a function of ω.

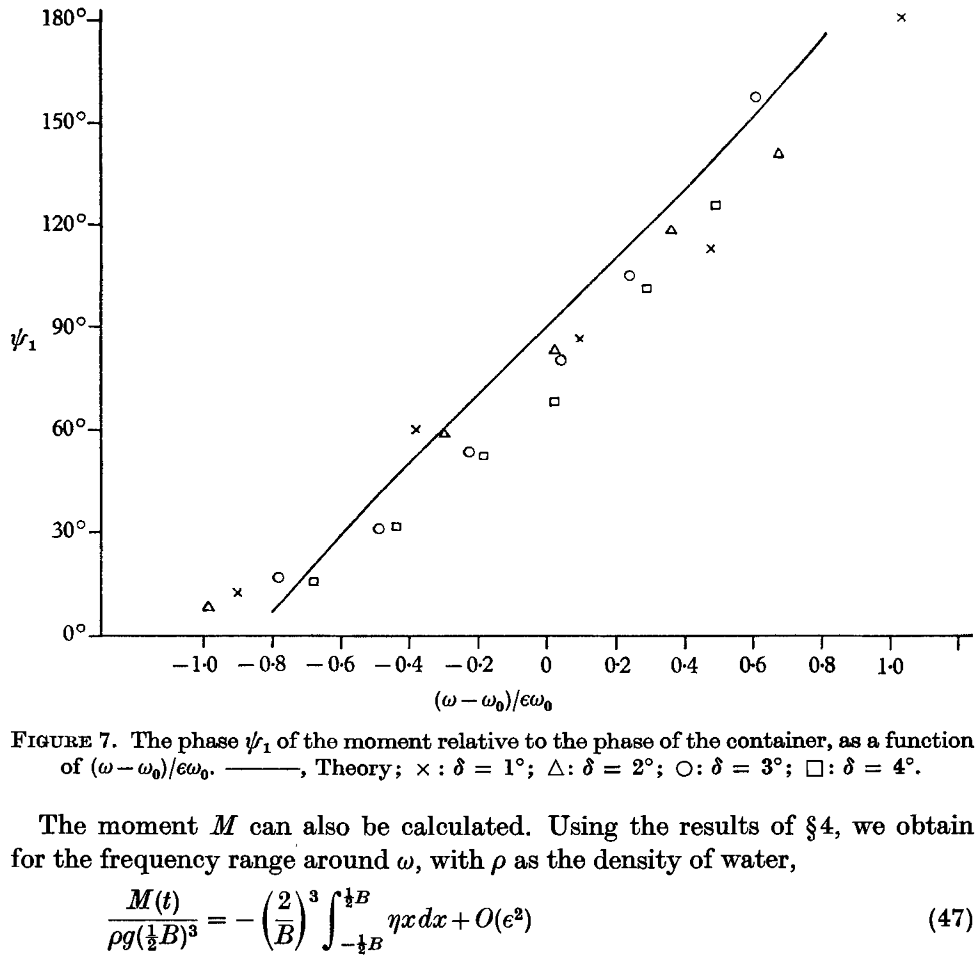

FIGURE 7. The phase ψ_1 of the moment relative to the phase of the container, as a function of $(\omega - \omega_0)/\epsilon\omega_0$. ———, Theory; ×: $\delta = 1°$; △: $\delta = 2°$; ○: $\delta = 3°$; □: $\delta = 4°$.

The moment M can also be calculated. Using the results of §4, we obtain for the frequency range around ω, with ρ as the density of water,

$$\frac{M(t)}{\rho g(\frac{1}{2}B)^3} = -\left(\frac{2}{B}\right)^3 \int_{-\frac{1}{2}B}^{\frac{1}{2}B} \eta x\, dx + O(\epsilon^2) \tag{47}$$

$$= \frac{128\epsilon h_0 A}{\pi^2 B}\left[(-1+\cos\tfrac{1}{2}\omega_0 t + \sin\tfrac{1}{2}\omega_0 t)\cos(\tfrac{1}{2}\phi + \tfrac{1}{4}\pi) + (-\tfrac{1}{2}\omega_0 t + \tfrac{1}{4}\pi + \sin\tfrac{1}{2}\omega_0 t - \cos\tfrac{1}{2}\omega_0 t)\sin(\tfrac{1}{2}\phi + \tfrac{1}{4}\pi)\right] + O(\epsilon^2) \tag{48}$$

for $0 < t < \pi/\omega$, and

$$\frac{128\epsilon h_0 A}{\pi^2 B}\left[(1+\cos\tfrac{1}{2}\omega_0 t - \sin\tfrac{1}{2}\omega_0 t)\cos(\tfrac{1}{2}\phi + \tfrac{1}{4}\pi) + (\tfrac{1}{2}\omega_0 t - \tfrac{3}{4}\pi + \sin\tfrac{1}{2}\omega_0 t + \cos\tfrac{1}{2}\omega_0 t)\sin(\tfrac{1}{2}\phi + \tfrac{1}{4}\pi)\right] + O(\epsilon^2) \tag{49}$$

for $\pi/\omega < t < 2\pi/\omega$. In the region of ω for conditions far from resonance the moment can be obtained from equation (8). Using $\omega t - \phi$ instead of ωt, we get from (8)

$$\frac{M(t)}{\rho g(\frac{1}{2}B)^3} = 8\delta\left(\frac{\omega_0}{\pi\omega}\right)^2 \sin(\omega t - \phi)\left\{\frac{2\omega_0}{\pi\omega}\tan\left(\frac{\pi\omega}{2\omega_0}\right) - 1\right\}. \tag{50}$$

The theoretical values of $\overline{M}_1$ for $\omega \ll \omega_0$ and for $\omega \gg \omega_0$ can be obtained from (50). These are the linear regions, as investigated by Binnie (1941).

For ω near ω_0 the value of $\overline{M}_1$ was obtained by expansion of the function given in (48) and (49) in a Fourier series of the type (46). For the first coefficient we find

$$(\tfrac{2}{3})^{\frac{3}{2}}\left(\frac{4}{\pi}\right)^4\left(\frac{\delta h_0}{B}\right)^{\frac{1}{2}}\left[1-\frac{B(\omega-\omega_0)^2}{32g\delta}\right]+O(\epsilon^2), \tag{51}$$

where use has been made of (12), (35) and (36). For ψ_1 we obtain

$$\psi_1 = -\phi - \arcsin\left[\frac{B(\omega-\omega_0)^2}{96g\delta - 3B(\omega-\omega_0)^2}\right]^{\frac{1}{2}} + O(\epsilon). \tag{52}$$

Expression (41) for ϕ can, with the aid of (35) and (36), be reduced to

$$\phi = -\tfrac{1}{2}\pi - 2\arcsin\left\{\frac{B(\omega-\omega_0)^2}{24g\delta}\right\}^{\frac{1}{2}} + O(\epsilon). \tag{53}$$

The theoretical values for ψ_1, obtained from (52) and (53) are given in figure 7 as a function of $(\omega-\omega_0)/\epsilon\omega_0 \approx (\omega-\omega_0)/(g\delta)^{\frac{1}{2}}$, together with the experimental values. For $\omega \ll \omega_0$ the moment is in phase with the oscillation of the container as shown by (50).

6. Discussion

Figures 4–7 show a good agreement between theory and experiment. In fact, the agreement is better than could be expected, if the following is considered. We recall that the theory is a first approximation in terms of $\epsilon = (\delta B/\pi h_0)^{\frac{1}{2}}$ (see equation (12)), so that differences of the order of ϵ^2 may be expected. A representative value of δ for the experiments is $\pi/60$. Then ϵ is as large as 0·5. Hence the agreement is, in general, surprisingly good and suggests that the coefficients of the terms in ϵ^2 are small.

Part of the discrepancy between theory and experiment is of course due to the neglect of viscosity in the theory. In particular, part of the difference between the measured phase of moment and the calculated phase, must be attributed to boundary-layer effects (cf. Chester (1964) where it is shown for the gas-dynamic case how viscosity changes the phase of the shock).

The results as given in § 4 are valid for frequencies which do not differ very much from the resonance frequency ω_0. In fact, it follows from consideration of the expression (53), for the phase of the hydraulic jump, that solutions including a jump exist only for $(\omega-\omega_0)^2 < 24g\delta/B$. This inequality can be written as

$$\left|1-\frac{\omega}{\omega_0}\right| < \left(\frac{24}{\pi}\right)^{\frac{1}{2}}\epsilon.$$

A similar result was obtained by Chester for the gas-dynamic case, using a quite different method. During the experiments we observed that at values of ω well beyond ω_0, the hydraulic jump disappeared, which is in agreement with the above result.

It may be of interest to note that at the disappearance of the jump a solitary wave was observed, which travelled back and forth between the walls of the container. This corresponds with the well-known fact (see Wehausen & Laitone 1960, §31) that the solitary wave travels at a speed slightly higher than the critical speed corresponding to a Froude number of unity. This solitary wave cannot be obtained from the present theory, since a higher-order shallow-water theory is needed, where in contrast to the first-order approximations (equations (5) and (6)) vertical accelerations are taken into account.

At high frequencies, $\omega \gg \omega_0$, a wave pattern represented by a solution of the type (8) reappeared.

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